NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 1

2021 Year 12 Course Assessment Task 4 (Trial Examination)
Thursday September 2, 2021

General instructions

- Working time – 2 hours.
  (plus 10 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer grid provided (on page 12)

SECTION II

- Commence each new question on a new booklet. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

NESA STUDENT #: .... **# BOOKLETS USED:** .....

Class (please ✔)

- ◯ 12MXX.1 – Mr Sekaran
- ◯ 12MXX.2 – Ms Ham
- ◯ 12MAX.1 – Mr Ho
- ◯ 12MAX.2 – Mrs Bhamra
- ◯ 12MAX.3 – Mr Lam
- ◯ 12MAX.4 – Mr Sun

Marker's use only.

QUESTION	1-10	11	12	13	14	Total	%
MARKS	$\overline{10}$	$\overline{16}$	$\overline{16}$	$\overline{14}$	$\overline{14}$	$\overline{70}$	

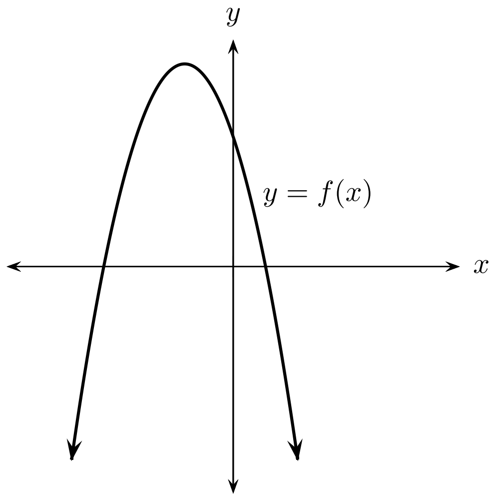
Section I

10 marks
Attempt Question 1 to 10
Allow approximately 15 minutes for this section

Mark your answers on the answer grid provided (labelled as page 12).

Questions **Marks**

1. The graph of $y = f(x)$ is shown. Consider the largest possible domain for which $y = f(x)$ has an inverse function $y = f^{-1}(x)$. **1**



How many points of intersection are there between $y = f(x)$ and $y = f^{-1}(x)$?

(A) 1 (C) 3

(B) 2 (D) 4

2. Which of the following has the same solutions as the inequality **1**

$$\frac{-x}{(x-3)(2x+1)} \leq \frac{1}{4}$$

(A) $(2x-3)(x+1)(x-3)(2x+1) \geq 0$ (C) $(2x-3)(x+1) \geq 0$

(B) $(2x-3)(x+1)(x-3)(2x+1) \leq 0$ (D) $(2x-3)(x+1) \leq 0$

Examination continues overleaf...

3. The letters in the word `CALCULUS` were randomly rearranged. **1**

What is the probability that `C` is at the start and end of the new arrangement?

(A) $\dfrac{1}{6}$

(B) $\dfrac{1}{14}$

(C) $\dfrac{1}{28}$

(D) $\dfrac{1}{56}$

4. Let $P(x) = x^3 + bx^2 + 6x + d$, where b and d are real numbers. **1**

Given that $(x-2)^2$ is a factor of $P(x)$, which of the following are possible values of b and d?

(A) $b = -6$, $d = 4$

(B) $b = 6$, $d = -4$

(C) $b = \dfrac{9}{2}$, $d = 2$

(D) $b = -\dfrac{9}{2}$, $d = -2$

5. Which of the following is the derivative of $y = -\cos^{-1}(\ln x)$? **1**

(A) $\dfrac{-1}{x\sqrt{1-2\ln x}}$

(B) $\dfrac{-1}{x\sqrt{(1+\ln x)(1-\ln x)}}$

(C) $\dfrac{1}{x\sqrt{1-2\ln x}}$

(D) $\dfrac{1}{x\sqrt{(1+\ln x)(1-\ln x)}}$

6. A curve is represented by the parametric equations $x = 5t^3 - 7t$ and $y = 2 + 3t - 4t^2$. **1**

Which of the following is an expression for $\dfrac{dy}{dx}$ in terms of t?

(A) $3 - 8t$

(B) $\dfrac{3-8t}{15t^2-7}$

(C) $(15t^2-7)(3-8t)$

(D) $\dfrac{15t^2-7}{3-8t}$

7. Which of the following is the unit vector perpendicular to $\underset{\sim}{p} = -6\underset{\sim}{i} + 2\underset{\sim}{j}$? **1**

(A) $\underset{\sim}{q} = \underset{\sim}{i} + 3\underset{\sim}{j}$

(B) $\underset{\sim}{q} = \dfrac{1}{\sqrt{10}}\underset{\sim}{i} + \dfrac{3}{\sqrt{10}}\underset{\sim}{j}$

(C) $\underset{\sim}{q} = \dfrac{1}{\sqrt{10}}\underset{\sim}{i} + \dfrac{-3}{\sqrt{10}}\underset{\sim}{j}$

(D) $\underset{\sim}{q} = \dfrac{-3}{\sqrt{10}}\underset{\sim}{i} + \dfrac{1}{\sqrt{10}}\underset{\sim}{j}$

Examination continues overleaf...

8. Which of the following is equivalent to $\displaystyle\int \sin 3x \sin 5x \, dx$? **1**

(A) $\dfrac{1}{16}\sin 8x - \dfrac{1}{4}\sin 2x + c$

(B) $\dfrac{1}{8}\sin 8x - \dfrac{1}{2}\sin 2x + c$

(C) $\dfrac{1}{4}\sin 2x - \dfrac{1}{16}\sin 8x + c$

(D) $\dfrac{1}{2}\sin 2x - \dfrac{1}{8}\sin 8x + c$

9. Which of the following integrals is equivalent to the integral I? Use the substitution $u = 3 - 2x^2$. **1**

$$I = \int_0^2 \frac{5x}{(3-2x^2)^4}\, dx$$

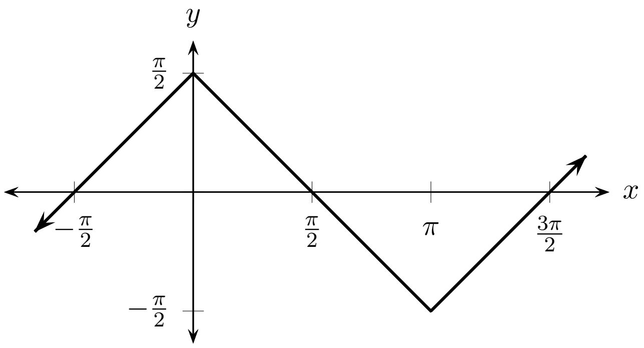
(A) $\dfrac{5}{4}\displaystyle\int_{-5}^{3} \frac{1}{u^4}\, du$

(B) $-\dfrac{5}{4}\displaystyle\int_{-5}^{3} \frac{1}{u^4}\, du$

(C) $\dfrac{5}{\sqrt{2}}\displaystyle\int_{-5}^{3} \frac{\sqrt{3-u}}{u^4}\, du$

(D) $-\dfrac{5}{\sqrt{2}}\displaystyle\int_{-5}^{3} \frac{\sqrt{3-u}}{u^4}\, du$

10. Which of the following equations is shown in the sketch below? **1**



(A) $y = \cos^{-1}(\sin x)$

(B) $y = \sin^{-1}(\cos x)$

(C) $y = \cos(\sin^{-1} x)$

(D) $y = \sin^{-1}(\sin x)$

Examination continues overleaf...

Section II

60 marks
Attempt Question 11 to 14
Allow approximately 1 hour 45 minutes for this section

Question 11 (16 Marks) Commence a NEW Writing Booklet **Marks**

(a) A class of fifteen mathematics students are seated around a circular table to discuss mathematical problems.

i. In how many different ways can the students be arranged? **1**

ii. If the seats are randomly assigned, find the probability that four particular students, Aidan, Christopher, Daniel and Matthew, are not all seated together as a group of four? Give your answer in the simplest form. **2**

(b) The polynomial equation $P(x) = ax^3 + 12x^2 + cx + d$ has roots α, β and γ, where a, c and d are real numbers.

i. Given that $\alpha + \beta + \gamma = 6$ and $\alpha^2 + \beta^2 + \gamma^2 = 32$, find the values of a and c. **2**

ii. Given that the polynomial can be expressed as $P(x) = (x+2)Q(x)$ where $Q(x)$ is a polynomial, find the value of d. **1**

(c) Using the substitution $t = \tan\theta$, solve for $0^\circ \leq \theta \leq 360^\circ$ **3**

$$2\sin 2\theta = 2 + 4\cos 2\theta$$

(d) The coefficients of the x^{17} and the x^{27} terms are equal in the expansion of **3**

$$\left(px^3 - \frac{q}{x^2}\right)^9$$

Prove that $p - 6q = 0$, where p and q are positive integers.

(e) i. Show that $\dfrac{1}{(n+2)!} - \dfrac{n+1}{(n+3)!} = \dfrac{2}{(n+3)!}$, where n is an integer. **1**

ii. Prove by mathematical induction that, for all positive integers n, **3**

$$\frac{1\times 2}{3!} + \frac{2\times 2^2}{4!} + \frac{3\times 2^3}{5!} + \cdots + \frac{n\times 2^n}{(n+2)!} = 1 - \frac{2^{n+1}}{(n+2)!}$$

Examination continues overleaf...

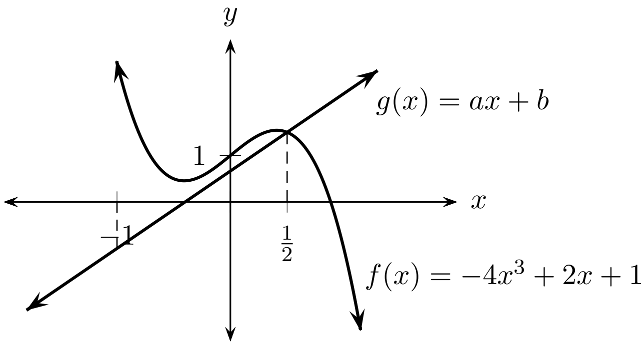
Question 12 (16 Marks) Commence a NEW Writing Booklet **Marks**

(a) Consider $f(x) = 4\cos^{-1}\left(\frac{1}{6}x\right) - \frac{\pi}{2}$.

i. State the domain and range of $f(x)$. **2**

ii. Hence sketch the graph of $f(x)$. (Do NOT find the x-intercepts) **2**

(b) The graphs of $f(x) = -4x^3 + 2x + 1$ and $g(x) = ax + b$ are shown.



i. Sketch the graph for $h(x) = -4\left|x\right|^3 + 2\left|x\right| + 1$. **1**

ii. The solution to the inequality $ax + b > -4\left|x\right|^3 + 2\left|x\right| + 1$ is **2**

$$x \in (-\infty, -1) \cup (\frac{1}{2}, \infty)$$

Find the values of a and b.

Examination continues overleaf…

(c) The position of jet ski relative to a stationary ship is plotted on the ship's radar. One unit on the radar represents a distance of 1 metre along the sea level, and the origin represents the ship's position.

The jet ski travels at a constant speed and is observed at point A on a position vector of $-20\underset{\sim}{i} + 36\underset{\sim}{j}$. One second later, the the jet ski is observed at point B on a position vector of $-2\underset{\sim}{i} + 51\underset{\sim}{j}$.

A crew member on the ship observes the motion of the jet ski using binoculars for ten seconds starting from point A, without taking his eyes off of the jet ski.

i. Find the position vector of the jet ski after the ten seconds. **2**

ii. Hence find the angle through which the crew member's binoculars moves through during this ten second interval, correct to the nearest minute. **2**

(d) i. Express $2\sqrt{3}\cos x - 2\sin x$ in the form $R\cos(x+\alpha)$, where $0 \le \alpha \le \frac{\pi}{2}$ and $R \ge 0$. **2**

ii. Hence sketch the graph of **3**

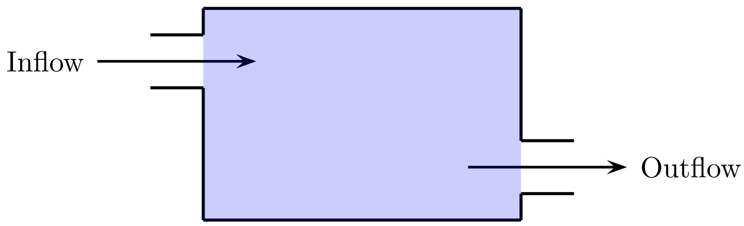
$$y = \frac{1}{2\sqrt{3}\cos x - 2\sin x} \quad \text{for } -\frac{\pi}{6} \le x \le \frac{11\pi}{6}$$

showing all important features.

Examination continues overleaf...

Question 13 (14 Marks) Commence a NEW Writing Booklet **Marks**

(a) A saltwater fish tank contains 250 litres of liquid. To keep the water clean, a pump draws out 15 litres per minute. To keep the amount of liquid in the tank constant, 15 litres of a saline solution with a salinity of 32.4 grams per litre is pumped into the tank every minute. Assume that the salt dissolves evenly into the solution in the tank.

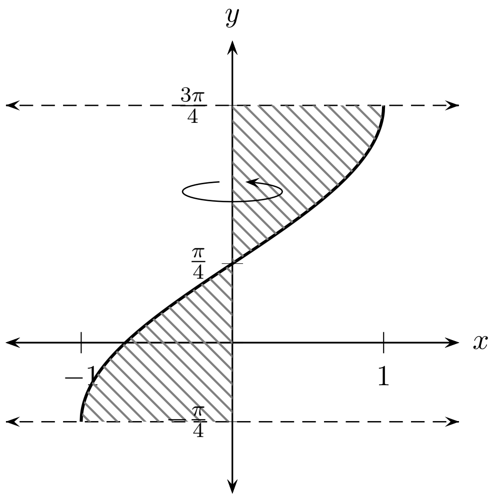


Let $Q(t)$ be the amount of dissolved salt in the fish tank after t minutes. The tank initially contains 8070 grams of salt dissolved in the liquid.

i. Show that $\dfrac{dQ}{dt} = \dfrac{24300 - 3Q}{50}$. **1**

ii. Hence solve the differential equation for the solution to the amount of salt $Q(t)$ in the tank at any time. **2**

(b) The region bounded by the curve $f(x) = \sin^{-1}(x) + \dfrac{\pi}{4}$ and the y-axis between the lines $y = -\dfrac{\pi}{4}$ and $y = \dfrac{3\pi}{4}$, is rotated one complete revolution about the y-axis.



i. Show that $\sin^2\left(x - \dfrac{\pi}{4}\right) = \dfrac{1}{2}(1 - \sin 2x)$. **2**

ii. Hence, or otherwise, find the volume of the solid of revolution. Give your answer as an exact value. **2**

Examination continues overleaf...

(c) The spread of a flu through a population is modelled by the equation

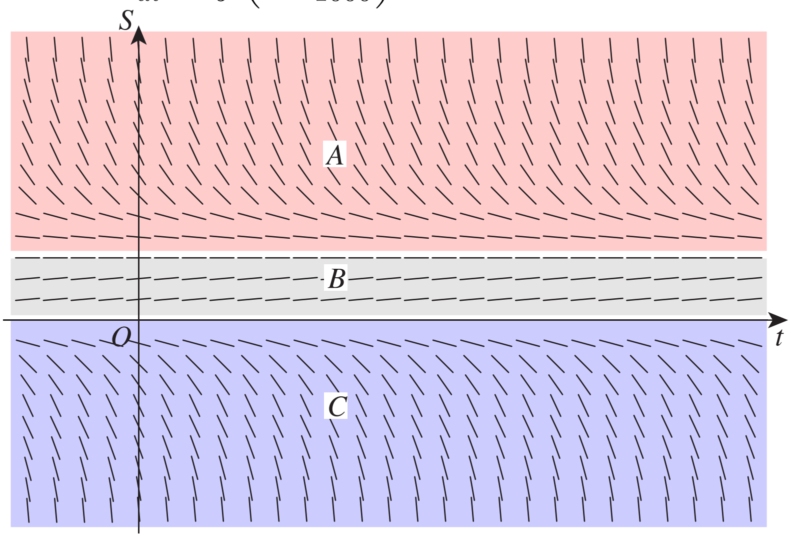
$$S = \frac{2000}{1 + 199e^{-0.4t}}$$

where S is the total number of people infected after t days.

i. Show that the given equation for S satisfies the differential equation **3**

$$\frac{dS}{dt} = \frac{S}{5}\left(2 - \frac{S}{1000}\right)$$

ii. The slope field of $\dfrac{dS}{dt} = \dfrac{S}{5}\left(2 - \dfrac{S}{1000}\right)$ is shown. **2**



The three regions A, B and C in which a solution curve can be found, have been labelled and shaded correspondingly.

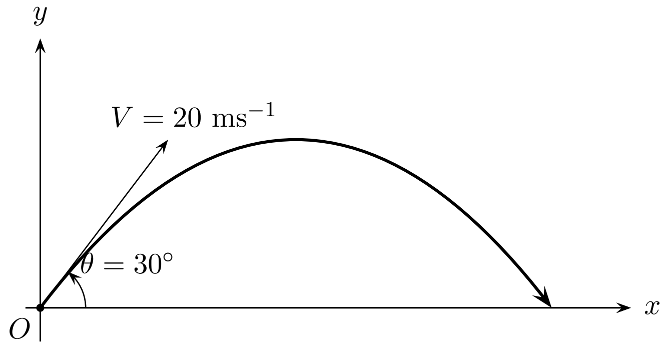
State which region the given solution curve S can exist in and justify your answer with reference to its constant solutions.

iii. Using this model, find on which day the rate of increase reaches a maximum. Give your answer correct to the nearest day. **2**

Examination continues overleaf...

Question 14 (14 Marks) Commence a NEW Writing Booklet **Marks**

(a) Eileen throws a rock from a fixed point O on level ground, with velocity $V = 20$ ms^{-1} at an angle of $\theta = 30°$ with the horizontal.



The flight path of the rock is given by

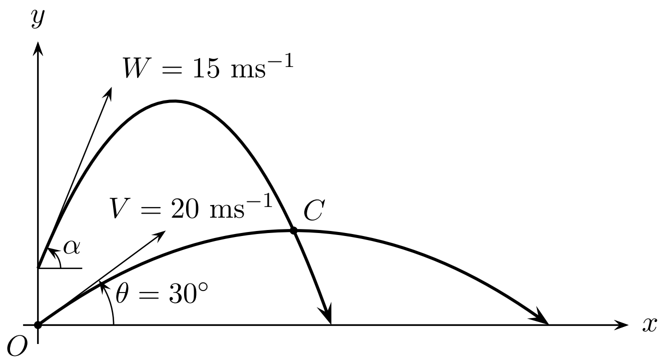
$$\begin{cases} x_r = Vt\cos\theta \\ y_r = -\dfrac{1}{2}gt^2 + Vt\sin\theta \end{cases}$$

(Do NOT prove this)

where t is the time in seconds after the stone was thrown. (Take $g = 10$ ms^{-2})

i. Find the maximum height that the rock reaches. **2**

Nathan is standing on jungle gym, 3 metres directly above the point O. Nathan throws the stick with velocity $W = 15$ ms^{-1} at angle α above the horizontal, such that it passes through point C where the stone reaches its maximum height.



ii. Prove that the position of stick is given by **3**

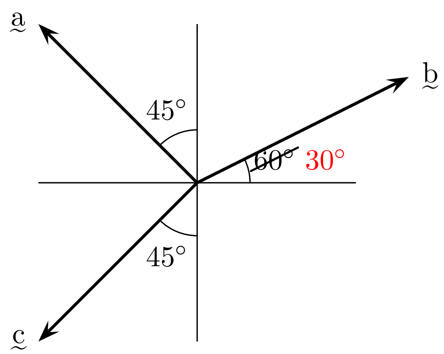
$$\underset{\sim}{\mathrm{r_s}} = \begin{pmatrix} 15t\cos\alpha \\ -5t^2 + 15t\sin\alpha + 3 \end{pmatrix}$$

(Take $g = 10$ ms^{-2})

iii. Determine whether the stick collides with the rock at point C. Justify your answer with calculations. **2**

Examination continues overleaf…

(b) A game of tug-of-war involves three teams who each attach their rope to a weight to pull in their direction as shown. Let $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ be the pulling force of Team A, Team B and Team C respectively.



Correction: the angle 60° was corrected to 30°.

It is given that $\left|\underset{\sim}{c}\right| = 200$ N. All teams are unable to move the weight from its starting position in any direction.

i. Show that **2**

$$\underset{\sim}{a} = \left(-\frac{1}{\sqrt{2}}\left|\underset{\sim}{a}\right|\right)\underset{\sim}{i} + \left(\frac{1}{\sqrt{2}}\left|\underset{\sim}{a}\right|\right)\underset{\sim}{j}$$

and

$$\cancel{\underset{\sim}{b} = \left(\frac{1}{2}\left|\underset{\sim}{b}\right|\right)\underset{\sim}{i} + \left(\frac{\sqrt{3}}{2}\left|\underset{\sim}{b}\right|\right)\underset{\sim}{j}}$$

$$\underset{\sim}{b} = \left(\frac{\sqrt{3}}{2}\left|\underset{\sim}{b}\right|\right)\underset{\sim}{i} + \left(\frac{1}{2}\left|\underset{\sim}{b}\right|\right)\underset{\sim}{j}$$

Correction: the component form of $\underset{\sim}{b}$ using the angle 30°.

ii. Briefly explain why $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = 0$. **1**

iii. Hence find the exact value of $\left|\underset{\sim}{a}\right|$ and $\left|\underset{\sim}{b}\right|$. **4**

End of paper.

Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

NESA STUDENT #: ....

Class (please ✔)

○ 12MXX.1 – Mr Sekaran ○ 12MAX.2 – Mrs Bhamra

○ 12MXX.2 – Ms Ham ○ 12MAX.3 – Mr Lam

○ 12MAX.1 – Mr Ho ○ 12MAX.4 – Mr Sun

Directions for multiple choice answers

- Read each question and its suggested answers.
- Select the alternative Ⓐ, Ⓑ, Ⓒ, or Ⓓ that best answers the question.
- Mark only one circle per question. There is only *one* correct choice per question.
- Fill in the response circle completely, using blue or black pen, e.g.

Ⓐ Ⓑ ● Ⓓ

- If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

Ⓐ Ⓑ ~~●~~ ●

- If you continue to change your mind, write the word `correct` and clearly indicate your final choice with an arrow as shown below:

Ⓐ Ⓑ ~~●~~ ~~●~~ correct

1 – Ⓐ Ⓑ Ⓒ Ⓓ

2 – Ⓐ Ⓑ Ⓒ Ⓓ

3 – Ⓐ Ⓑ Ⓒ Ⓓ

4 – Ⓐ Ⓑ Ⓒ Ⓓ

5 – Ⓐ Ⓑ Ⓒ Ⓓ

6 – Ⓐ Ⓑ Ⓒ Ⓓ

7 – Ⓐ Ⓑ Ⓒ Ⓓ

8 – Ⓐ Ⓑ Ⓒ Ⓓ

9 – Ⓐ Ⓑ Ⓒ Ⓓ

10 – Ⓐ Ⓑ Ⓒ Ⓓ

Suggested Band 6 Responses

1. (A) **2.** (A) **3.** (C) **4.** (D) **5.** (D)
6. (B) **7.** (B) **8.** (C) **9.** (A) **10.** (B)

Question 11 (Ho)

(a) i. (1 mark)

✓ [1] for final answer.

$$\begin{aligned}\text{Ways} &= (15-1)! \\ &= 14!\end{aligned}$$

ii. (2 marks)

✓ [1] for correct number of arrangements.

✓ [1] for final answer.

Let E be the event where Aidan, Christopher, Daniel and Matthew are seated together.

$$\begin{aligned}P(E) &= \frac{(12-1)! \times 4!}{14!} \\ &= \frac{1}{91} \\ P(\overline{E}) &= 1 - \frac{1}{91} \\ &= \frac{90}{91}\end{aligned}$$

(b) i. (2 marks)

✓ [1] for correct value of a.

✓ [1] for correct value of c.

$$\begin{aligned}\alpha+\beta+\gamma &= -\frac{12}{a} \\ 6 &= -\frac{12}{a} \\ a &= -2\end{aligned}$$

$$\begin{aligned}(\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma) &= \alpha^2+\beta^2+\gamma^2 \\ 36 - 2\left(\frac{c}{-2}\right) &= 32 \\ c &= -4\end{aligned}$$

$\therefore a = -2$ and $c = -4$.

ii. (2 marks)

✓ [1] for final answer.

$(x+2)$ is a factor of $P(x)$:

$$\begin{aligned}P(-2) &= 0 \\ -8(-2) + 48 - 2(-4) + d &= 0 \\ \therefore d &= -72\end{aligned}$$

(c) (3 marks)

✓ [1] for correct quadratic equation for t.

✓ [1] for one correct pair of solutions for θ.

✓ [1] for all correct solutions for θ.

Using $t = \tan\theta$:

$$\begin{aligned}2\left(\frac{2t}{1+t^2}\right) &= 2 + 4\left(\frac{1-t^2}{1+t^2}\right) \\ 4t &= (2+2t^2) + (4-4t^2) \\ 2t^2 + 4t - 6 &= 0 \\ t^2 + 2t - 3 &= 0 \\ (t+3)(t-1) &= 0\end{aligned}$$

$$\begin{aligned}t = -3 &\text{ or } t = 1 \\ \tan\theta = -3 &\text{ or } \tan\theta = 1\end{aligned}$$

If $\tan\theta = -3$: ref $\angle = 71°34'$
$\theta = 108°26'$ or $288°26'$

If $\tan\theta = 1$: ref $\angle = 45°$
$\theta = 45°$ or $225°$

$\therefore \theta = 45°,\ 108°26',\ 225°$ or $288°26'$

(d) (3 marks)

✓ [1] for correct binomial expansion.
✓ [1] for correct expressions for c_0 and c_2.
✓ [1] for final result.

$$\begin{aligned}\left(px^3 - \frac{q}{x^2}\right)^9 &= \sum_{r=0}^{9} \binom{9}{r} \left(px^3\right)^{9-r} \left(-9x^{-2}\right)^r \\ &= \sum_{r=0}^{9} \binom{9}{r} (-1)^r p^{9-r} q^r x^{27-5x}\end{aligned}$$

For the x^{17} term:

$$\begin{aligned}27 - 5r &= 17 \\ r &= 2 \\ c_2 &= \binom{9}{2} p^7 q^2\end{aligned}$$

For the x^{27} term:

$$\begin{aligned}27 - 5r &= 27 \\ r &= 0 \\ c_0 &= \binom{9}{0} p^9\end{aligned}$$

Given $c_0 = c_2$:

$$\begin{aligned}(1)p^9 &= (36)p^7 q^2 \\ p^9 - 36p^2 q^2 &= 0 \\ p^7(p^2 - 36q^2) &= 0 \\ p^7(p - 6q)(p + 6q) &= 0\end{aligned}$$

$$p = 0,\ p - 6q = 0 \text{ or } p + 6q = 0$$

But given $p > 0$ and $q > 0$.

Hence $p - 6q = 0$ only.

(e) i. (1 mark)

✓ [1] for final proof.

$$\begin{aligned}\text{LHS} &= \frac{1}{(n+2)!} - \frac{(n+1)}{(n+3)(n+2)!} \\ &= \frac{(n+3) - (n+1)}{(n+3)(n+2)!} \\ &= \frac{2}{(n+3)!}\end{aligned}$$

$\therefore$ LHS = RHS

ii. (3 marks)

✓ [1] for proving the base case
✓ [1] for applying the inductive step
✓ [1] for final proof

Base case: Prove true for $n = 1$.

$$\begin{aligned}\text{LHS} &= \frac{1 \times 2}{(1+2)!} \\ &= \frac{1}{3} \\ \text{RHS} &= 1 - \frac{2^{1+1}}{(1+2)!} \\ &= \frac{1}{3}\end{aligned}$$

$\therefore$ LHS = RHS, and the statement is true for $n = 1$.

Inductive step:
Assume true for $n = k$.

$$\frac{1 \times 2}{3!} + \frac{2 \times 2^2}{4!} + \frac{3 \times 2^3}{5!} + \cdots + \frac{k \times 2^k}{(k+2)!} = 1 - \frac{2^{k+1}}{(k+2)!}$$

Prove true for $n = k+1$:

RTP: $\dfrac{1\times 2}{3!} + \dfrac{2\times 2^2}{4!} + \dfrac{3\times 2^3}{5!} + \cdots + \dfrac{(k+1)\times 2^{k+1}}{(k+3)!} = 1 - \dfrac{2^{k+2}}{(k+3)!}$

$$\text{LHS} = \frac{1\times 2}{3!} + \frac{2\times 2^2}{4!} + \frac{3\times 2^3}{5!} + \cdots + \frac{k\times 2^k}{(k+2)!} + \frac{(k+1)\times 2^{k+1}}{(k+3)!}$$

By the inductive step:

$$\text{LHS} = \left(1 - \frac{2^{k+1}}{(k+2)!}\right) + \frac{(k+1)\times 2^{k+1}}{(k+3)!}$$

$$= 1 - \frac{(k+3)\times 2^{k+1}}{(k+3)!} + \frac{(k+1)\times 2^{k+1}}{(k+3)!}$$

$$= 1 + \frac{[-(k+3)+(k+1)]\times 2^{k+1}}{(k+3)!}$$

$$= 1 + \frac{(-2)\times 2^{k+1}}{(k+3)!}$$

$$= 1 - \frac{2^{k+2}}{(k+3)!}$$

$\therefore$ LHS = RHS, and the statement is true for $n = k+1$.

Hence by mathematical induction, the statement is true for all positive integers n.

Question 12 (Sun)

(a) i. (2 marks)

✓ [1] for correct domain.
✓ [1] for correct range.

$$-1 \le \frac{1}{6}x \le 1$$
$$-6 \le x \le 6$$

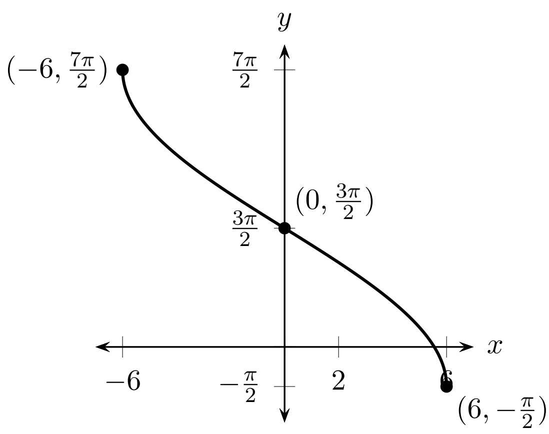
$\therefore$ Domain is $-6 \le x \le 6$

$$0 \le \cos^{-1}\theta \le \pi$$
$$0 \le 4\cos^{-1}\theta \le 4\pi$$
$$-\frac{\pi}{2} \le 4\cos^{-1}\theta - \frac{\pi}{2} \le \frac{7\pi}{2}$$

$\therefore$ Range is $-\frac{\pi}{2} \le y \le \frac{7\pi}{2}$

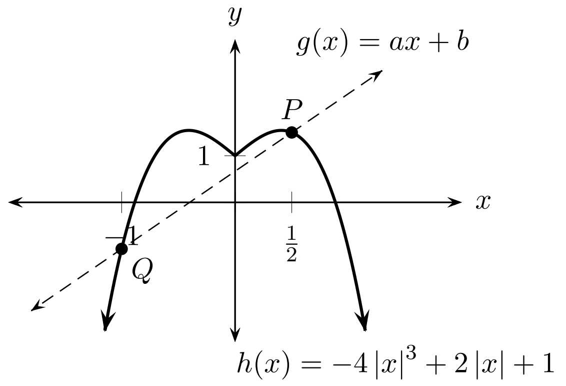
ii. (2 marks)

✓ [1] for shape
✓ [1] for endpoints and y-intercept



(b) i. (1 mark)

✓ [1] for final sketch of $h(x)$.



ii. (2 marks)

✓ [1] for correct value of a.

✓ [1] for correct value of b.

P and Q are the points of intersection of $g(x)$ and $h(x)$.

$$h\left(\frac{1}{2}\right) = -4\left|\frac{1}{2}\right|^3 + 2\left|\frac{1}{2}\right| + 1 = \frac{3}{2}$$

$$\therefore P\left(\frac{1}{2}, \frac{3}{2}\right)$$

$$h(-1) = -4|-1|^3 + 2|-1| + 1 = -1$$

$$\therefore Q(-1,-1)$$

$$a = \frac{-1-\frac{3}{2}}{-1-\frac{1}{2}} = \frac{5}{3}$$

Substitute $Q(-1,-1)$ into $g(x) = \frac{5}{3}x + b$:

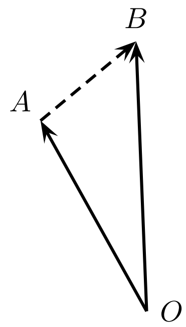
$$-1 = \frac{5}{3}(-1) + b$$
$$b = \frac{2}{3}$$

$\therefore a = \frac{5}{3}$ and $b = \frac{2}{3}$.

(c) i. (2 marks)

✓ [1] for correct vector for $\overrightarrow{AB}$

✓ [1] for correct position vecor $\overrightarrow{OC}$



$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -2 \\ 51 \end{pmatrix} - \begin{pmatrix} -20 \\ 36 \end{pmatrix} = \begin{pmatrix} 18 \\ 15 \end{pmatrix}$$

Let C be the position of the jet ski after 10 seconds.

$$\overrightarrow{OC} = \overrightarrow{OA} + 10\overrightarrow{AB} = \begin{pmatrix} -20 \\ 36 \end{pmatrix} + 10\begin{pmatrix} 18 \\ 15 \end{pmatrix} = \begin{pmatrix} 160 \\ 186 \end{pmatrix}$$

Hence the position vector after 10 seconds is $160\underset{\sim}{i} + 186\underset{\sim}{j}$.

ii. (2 marks)

✓ [1] for substitution into formula

✓ [1] for correct value for θ

Let θ be the angle.

$$\overrightarrow{OA} \cdot \overrightarrow{OC} = \left|\overrightarrow{OA}\right|\left|\overrightarrow{OC}\right| \cos\theta$$

$$\begin{pmatrix} -20 \\ 36 \end{pmatrix} \cdot \begin{pmatrix} 160 \\ 186 \end{pmatrix} = \left(\sqrt{(-20)^2 + 36^2}\right)\left(\sqrt{160^2 + 186^2}\right)\cos\theta$$

$$3496 = (\sqrt{1696})(\sqrt{60196})\cos\theta$$

$$\cos\theta = \frac{3496}{(\sqrt{1696})(\sqrt{60196})}$$

$$\therefore \theta = 69°45'$$

Hence the angle through which the crew member's binoculars moves is $69°45'$.

(d) i. (2 marks)

✓ [1] for correct value for R

✓ [1] for correct value for α

$2\sqrt{3}\cos x - 2\sin x = R\cos(x+\alpha)$.

$$= R(\cos x\cos\alpha - \sin x\sin\alpha)$$

Comparing the coefficients of $\cos x$:

$$2\sqrt{3} = R\cos\alpha \qquad (12.1)$$

Comparing the coefficeints of $\sin x$:

$$2 = R\sin\alpha \qquad (12.2)$$

Equations $(12.1)^2 + (12.2)^2$:

$$\begin{aligned} R^2(\sin^2\alpha + \cos^2\alpha) &= 12 + 4 \\ R^2 &= 16 \\ \text{since } R \geq 0: \quad R &= 4 \end{aligned}$$

Equations $(12.2) \div (12.1)$:

$$\begin{aligned} \tan\alpha &= \frac{1}{\sqrt{3}} \\ \alpha &= \frac{\pi}{6} \end{aligned}$$

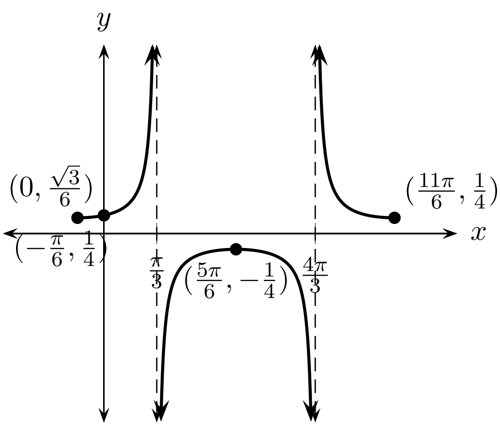
$\therefore 2\sqrt{3}\cos x - 2\sin x = 4\cos(x + \frac{\pi}{6})$.

ii. (3 marks)

✓ [1] for correct asymptotes

✓ [1] for correct turning points

✓ [1] for correct shape



Question 13 (Lam)

(a) i. (1 mark)

✓ [1] for final proof

Rate of inflow = 486 g/min

Rate of outflow = $\frac{Q}{250} \times 15$ g/min

$$\begin{aligned} \frac{dQ}{dt} &= 486 - \frac{15Q}{250} \\ &= \frac{24300}{50} - \frac{3Q}{50} \\ \therefore \frac{dQ}{dt} &= \frac{24300 - 3Q}{50} \end{aligned}$$

ii. (2 marks)

✓ [1] for correct integral

✓ [1] for final result

$$\begin{aligned} \int \frac{1}{24300 - 3Q}\,dQ &= \int \frac{1}{50}\,dt \\ -\frac{1}{3}\ln(24300 - 3Q) + c_1 &= \frac{1}{50}t + c_2 \\ \ln(24300 - 3Q) &= -\frac{3}{50}t + c \\ 24300 - 3Q &= e^{-\frac{3}{50}t + c} \\ -3Q &= Ae^{-\frac{3}{50}t} - 24300 \\ Q &= Be^{-\frac{3}{50}t} + 8100 \end{aligned}$$

where B is some constant

At $t = 0$, $Q = 8070$:

$$\begin{aligned} 8070 &= B + 8100 \\ B &= -30 \end{aligned}$$

$\therefore Q(t) = 8100 - 30e^{-\frac{3}{50}t}$

(b) i. (2 marks)

✓ [1] for significant progress

✓ [1] for final proof

$$\begin{aligned}\text{LHS} &= \left(\sin x\cos\frac{\pi}{4} - \cos x\sin\frac{\pi}{4}\right)^2\\ &= \frac{1}{2}(\sin x - \cos x)^2\\ &= \frac{1}{2}(\sin^2 x + \cos^2 x - 2\sin x\cos x)\\ &= \frac{1}{2}(1-\sin 2x)\end{aligned}$$

$\therefore \text{LHS} = \text{RHS}$

ii. (2 marks)

✓ [1] for correct integral

✓ [1] for correct exact volume

$$\begin{aligned}y &= \sin^{-1}(x) - \frac{\pi}{4}\\ y + \frac{\pi}{4} &= \sin^{-1}(x)\\ x &= \sin\left(y + \frac{\pi}{4}\right)\end{aligned}$$

Let V be the volume of the solid of revolution about the y-axis:

$$V = \pi\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \left[\sin\left(y - \frac{\pi}{4}\right)\right]^2\,dy$$

From part (i):

$$\begin{aligned}&= \pi^2\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{2}(1-\sin 2x)\,dy\\ &= \frac{\pi^2}{2}\left[x + \frac{1}{2}\cos 2x\right]_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\\ &= \frac{\pi^2}{2}\left[\left(\frac{3\pi}{4} + \frac{1}{2}\cos\frac{3\pi}{2}\right)\right.\\ &\qquad\left. - \left(-\frac{\pi}{4} + \frac{1}{2}\cos\left(-\frac{\pi}{2}\right)\right)\right]\\ &= \frac{\pi^2}{2}\left[\left(\frac{3\pi}{4}+0\right) - \left(-\frac{\pi}{4}+0\right)\right]\end{aligned}$$

$$\therefore V = \frac{\pi^2}{2}$$

(c) i. (3 marks)

✓ [1] for correct derivative

✓ [1] for significant progress

✓ [1] for final proof

$$\begin{aligned}\frac{dS}{dt} &= \frac{-2000(-0.4\times 199e^{-0.4t})}{(1+199e^{-0.4t})^2}\\ &= \frac{2000}{1+199e^{-0.4t}}\left(\frac{2}{5}\times\frac{199e^{-0.4t}}{1+199e^{-0.4t}}\right)\\ &= \frac{S}{5}\times\frac{2(1+199e^{-0.4t}-1)}{1+199e^{-0.4t}}\\ &= \frac{S}{5}\times\frac{2(1+199e^{-0.4t})-2}{1+199e^{-0.4t}}\\ &= \frac{S}{5}\left(2 - \frac{1}{1000}\times\frac{2000}{1+199e^{-0.4t}}\right)\end{aligned}$$

$$\therefore \frac{dS}{dt} = \frac{S}{5}\left(2 - \frac{S}{1000}\right)$$

ii. (2 marks)

✓ [1] for correct constant solutions

✓ [1] for correct region

From part (i), when $\frac{dS}{dt} = 0$:

$$\frac{S}{5}\left(2 - \frac{S}{1000}\right) = 0$$

$$S = 0 \text{ or } S = 2000$$

$\therefore$ The logistic curve S is bounded by the two constant solutions $S = 0$ and $S = 2000$.

When $t = 0$:

$$\begin{aligned}S &= \frac{2000}{1+199e^0}\\ &= 10\end{aligned}$$

The initial value $S = 10$ is within the interval $[0, 2000]$.

Hence, the solution curve S is in region B.

iii. (2 marks)

✓ [1] for correct value of S

✓ [1] for final integer solution

The graph of $\frac{dS}{dt}$ is a concave down parabola with roots $S = 0$ and $S = 2000$.

$$\begin{aligned} S_{\text{vertex}} &= \frac{0 + 2000}{2} \\ &= 1000 \end{aligned}$$

When $S = 1000$:

$$1000 = \frac{2000}{1 + 199e^{-0.4t}}$$

$$\begin{aligned} 1 + 199e^{-0.4t} &= 2 \\ e^{-0.4t} &= \frac{1}{199} \end{aligned}$$

$$\begin{aligned} t &= -\frac{5}{2}\ln\left(\frac{1}{199}\right) \\ &= 13.233... \end{aligned}$$

Hence the rate of increase is a maximum on the 14th day.

Question 14 (Bhamra)

(a) i. (2 marks)

✓ [1] for correct value of t

✓ [1] for correct maximum height

$$\begin{aligned} \dot{y_r} &= -gt + V\sin\theta \\ &= -10t + 20\sin 30^\circ \\ &= -10t + 10 \end{aligned}$$

Maximum height occurs when $\dot{y_r} = 0$:

$$\begin{aligned} 0 &= -10t + 10 \\ t &= 1 \end{aligned}$$

When $t = 1$:

$$\begin{aligned} y_r &= -\frac{1}{2}(10)(1)^2 + 20(1)\sin 30^\circ \\ &= 5 \end{aligned}$$

$\therefore$ The maximum height of the rock is 5 metres.

ii. (3 marks)

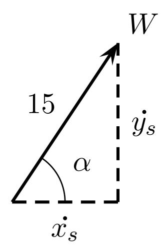
✓ [1] for correct $\underset{\sim}{\mathrm{a_s}}$

✓ [1] for correct $\underset{\sim}{\mathrm{v_s}}$

✓ [1] for final proof

$$\underset{\sim}{\mathrm{a_s}}(t) = \begin{pmatrix} 0 \\ -10 \end{pmatrix}$$

$$\underset{\sim}{\mathrm{v_s}}(t) = \begin{pmatrix} c_1 \\ -10t + c_2 \end{pmatrix}$$



When $t = 0$:

$$\dot{x_s} = 15\cos\alpha$$
$$\dot{y_s} = 15\sin\alpha$$

$$\therefore \underset{\sim}{\mathrm{v_s}}(t) = \begin{pmatrix} 15\cos\alpha \\ -10t + 15\sin\alpha \end{pmatrix}$$

$$\underset{\sim}{\mathrm{r_s}}(t) = \begin{pmatrix} 15t\cos\alpha + c_3 \\ -5t^2 + 15t\sin\alpha + c_4 \end{pmatrix}$$

When $t = 0$, $x = 0$ and $y = 3$.

$$\therefore \underset{\sim}{\mathrm{r_s}}(t) = \begin{pmatrix} 15t\cos\alpha \\ -5t^2 + 15t\sin\alpha + 3 \end{pmatrix}$$

iii. (2 marks)

✓ [1] for appropriate calculations

✓ [1] for correct conclusion

For the rock at point C:
From part (i), $t = 1$.

$$\begin{aligned} x_r &= 20(1)\cos 30^\circ \\ &= 10\sqrt{3} \end{aligned}$$

If the stick collides with the rock at point C:

$$\text{when } t = 1,\ x_s = 10\sqrt{3}$$

From part (ii), $x = 15t\cos\alpha$:

$$\begin{aligned} 10\sqrt{3} &= 15(1)\cos\alpha \\ \cos\alpha &= \frac{2\sqrt{3}}{3} \end{aligned}$$

But $-1 \le \cos\alpha \le 1$

$\therefore$ There is no such value of α for which the the horizontal displacement of the stick is $x_s = 10\sqrt{3}$ when $t = 1$.

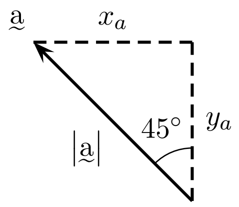
Hence the rock and stick do not collide at C since they reach the horizontal position of C at different times.

(b) i. (2 marks)

✓ [1] for component form of $\underset{\sim}{a}$

✓ [1] for component form of $\underset{\sim}{b}$

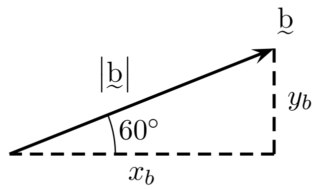
The corrected solutions of Question 14 (b) using the angle 30° is provided in red text within the next section.



$$\begin{aligned} x_a &= -\left|\underset{\sim}{a}\right|\cos 45^\circ \\ &= -\frac{1}{\sqrt{2}}\left|\underset{\sim}{a}\right| \end{aligned}$$

$$\begin{aligned} y_a &= \frac{1}{\sqrt{2}}\left|\underset{\sim}{a}\right| \\ &= \frac{}{\sqrt{2}}\left|\underset{\sim}{a}\right| \end{aligned}$$

$$\therefore \underset{\sim}{a} = \left(-\frac{1}{\sqrt{2}}\left|\underset{\sim}{a}\right|\right)\underset{\sim}{i} + \left(\frac{1}{\sqrt{2}}\left|\underset{\sim}{a}\right|\right)\underset{\sim}{j}$$



$$\begin{aligned} x_b &= \left|\underset{\sim}{b}\right|\cos 60^\circ \\ &= \frac{1}{2}\left|\underset{\sim}{b}\right| \end{aligned}$$

$$\begin{aligned} y_b &= \left|\underset{\sim}{b}\right|\sin 60^\circ \\ &= \frac{\sqrt{3}}{2}\left|\underset{\sim}{b}\right| \end{aligned}$$

$$\therefore \underset{\sim}{b} = \left(\frac{1}{2}\left|\underset{\sim}{b}\right|\right)\underset{\sim}{i} + \left(\frac{\sqrt{3}}{2}\left|\underset{\sim}{b}\right|\right)\underset{\sim}{j}$$

ii. (1 mark)

✓ [1] for correct explanation

The forces of vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are in equilibrium as the weight's motion remains constant at rest.

Hence the resultant force is zero:

$$\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = 0$$

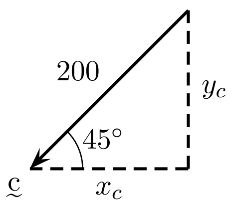
iii. (4 marks)

✓ [1] for component form of $\underset{\sim}{c}$

✓ [1] for simultaneous equations

✓ [1] for correct values of $\left|\underset{\sim}{a}\right|$

✓ [1] for correct values of $\left|\underset{\sim}{b}\right|$



$$\begin{aligned} x_c &= -200\cos 45^\circ \\ &= -100\sqrt{2} \end{aligned}$$

$$\begin{aligned} y_c &= -200\sin 45^\circ \\ &= -100\sqrt{2} \end{aligned}$$

$$\underset{\sim}{c} = (-100\sqrt{2})\,\underset{\sim}{i} + (-100\sqrt{2})\,\underset{\sim}{j}$$

From part (ii):

$$\begin{pmatrix} -\frac{1}{\sqrt{2}}\left|\underset{\sim}{a}\right| \\ \frac{1}{\sqrt{2}}\left|\underset{\sim}{a}\right| \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\left|\underset{\sim}{b}\right| \\ \frac{\sqrt{3}}{2}\left|\underset{\sim}{b}\right| \end{pmatrix} + \begin{pmatrix} -100\sqrt{2} \\ -100\sqrt{2} \end{pmatrix} = 0$$

$$\therefore \frac{-1}{\sqrt{2}}\left|\underset{\sim}{a}\right| + \frac{1}{2}\left|\underset{\sim}{b}\right| - 100\sqrt{2} = 0 \qquad (14.1)$$

$$\frac{1}{\sqrt{2}}\left|\underset{\sim}{a}\right| + \frac{\sqrt{3}}{2}\left|\underset{\sim}{b}\right| - 100\sqrt{2} = 0 \qquad (14.2)$$

Equation (14.1) + (14.2):

$$\begin{aligned} \frac{1}{2}\left|\underset{\sim}{b}\right|(1+\sqrt{3}) - 200\sqrt{2} &= 0 \\ \left|\underset{\sim}{b}\right| &= \frac{400\sqrt{2}}{1+\sqrt{3}} \\ &= \frac{400\sqrt{2}(1-\sqrt{3})}{1^2-(\sqrt{3})^2} \\ &= \frac{400(\sqrt{2}-\sqrt{6})}{-2} \end{aligned}$$

$$\therefore \left|\underset{\sim}{b}\right| = 200(\sqrt{6}-\sqrt{2}) \quad N \quad (14.3)$$

Substitute equation (14.3) into (14.1):

$$-\frac{1}{\sqrt{2}}\left|\underset{\sim}{a}\right| + \frac{1}{2} \times 200(\sqrt{6}-\sqrt{2}) - 100\sqrt{2} = 0$$

$$\frac{1}{\sqrt{2}}\left|\underset{\sim}{a}\right| = 100(\sqrt{6}-2\sqrt{2})$$

$$\left|\underset{\sim}{a}\right| = 100(2\sqrt{3}-4)$$

$$\therefore \left|\underset{\sim}{a}\right| = 200(\sqrt{3}-2) \quad N$$

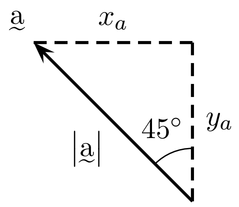
Corrected solution for Question 14 (b)

(Bhamra)

b i. (2 marks)

✓ [1] for component form of $\underset{\sim}{a}$

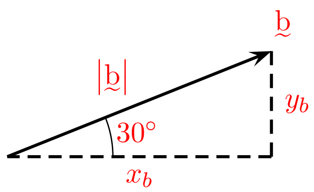
✓ [1] for component form of $\underset{\sim}{b}$



$$\begin{aligned} x_a &= -\left|\underset{\sim}{a}\right| \cos 45° \\ &= -\frac{1}{\sqrt{2}}\left|\underset{\sim}{a}\right| \end{aligned}$$

$$\begin{aligned} y_a &= \frac{1}{\sqrt{2}}\left|\underset{\sim}{a}\right| \\ &= \frac{}{\sqrt{2}}\left|\underset{\sim}{a}\right| \end{aligned}$$

$$\therefore \underset{\sim}{a} = \left(-\frac{1}{\sqrt{2}}\left|\underset{\sim}{a}\right|\right)\underset{\sim}{i} + \left(\frac{1}{\sqrt{2}}\left|\underset{\sim}{a}\right|\right)\underset{\sim}{j}$$



$$\begin{aligned} x_b &= \left|\underset{\sim}{b}\right| \cos 30° \\ &= \frac{\sqrt{3}}{2}\left|\underset{\sim}{b}\right| \end{aligned}$$

$$\begin{aligned} y_b &= \left|\underset{\sim}{b}\right| \sin 30° \\ &= \frac{1}{2}\left|\underset{\sim}{b}\right| \end{aligned}$$

$$\therefore \underset{\sim}{b} = \left(\frac{\sqrt{3}}{2}\left|\underset{\sim}{b}\right|\right)\underset{\sim}{i} + \left(\frac{1}{2}\left|\underset{\sim}{b}\right|\right)\underset{\sim}{j}$$

ii. (1 mark)

✓ [1] for correct explanation

The forces of vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are in equilibrium as the weight's motion remains constant at rest.

Hence the resultant force is zero:

$$\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = 0$$

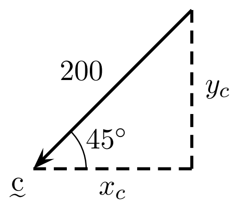
iii. (4 marks)

✓ [1] for component form of $\underset{\sim}{c}$

✓ [1] for simultaneous equations

✓ [1] for correct values of $\left|\underset{\sim}{a}\right|$

✓ [1] for correct values of $\left|\underset{\sim}{b}\right|$



$$\begin{aligned} x_c &= -200 \cos 45° \\ &= -100\sqrt{2} \end{aligned}$$

$$\begin{aligned} y_c &= -200 \sin 45° \\ &= -100\sqrt{2} \end{aligned}$$

$$\underset{\sim}{c} = (-100\sqrt{2})\,\underset{\sim}{i} + (-100\sqrt{2})\,\underset{\sim}{j}$$

From part (ii):

$$\begin{pmatrix} -\frac{1}{\sqrt{2}}\left|\underset{\sim}{a}\right| \\ \frac{1}{\sqrt{2}}\left|\underset{\sim}{a}\right| \end{pmatrix} + \begin{pmatrix} \frac{\sqrt{3}}{2}\left|\underset{\sim}{b}\right| \\ \frac{1}{2}\left|\underset{\sim}{b}\right| \end{pmatrix} + \begin{pmatrix} -100\sqrt{2} \\ -100\sqrt{2} \end{pmatrix} = 0$$

$$\therefore \frac{-1}{\sqrt{2}}\left|\underset{\sim}{a}\right| + \frac{\sqrt{3}}{2}\left|\underset{\sim}{b}\right| - 100\sqrt{2} = 0 \tag{14.4}$$

$$\frac{1}{\sqrt{2}}\left|\underset{\sim}{a}\right| + \frac{1}{2}\left|\underset{\sim}{b}\right| - 100\sqrt{2} = 0 \tag{14.5}$$

Equation (14.4) + (14.5):

$$\frac{1}{2}\left|\underset{\sim}{b}\right|(\sqrt{3}+1) - 200\sqrt{2} = 0$$

$$\left|\underset{\sim}{b}\right| = \frac{400\sqrt{2}}{\sqrt{3}+1}$$

$$= \frac{400\sqrt{2}(\sqrt{3}-1)}{(\sqrt{3})^2 - 1^2}$$

$$= \frac{400(\sqrt{6}-\sqrt{2})}{2}$$

$$\therefore \left|\underset{\sim}{b}\right| = 200(\sqrt{6}-\sqrt{2}) \quad N \quad (14.6)$$

Substitute equation (14.6) into (14.4):

$$\frac{-1}{\sqrt{2}}\left|\underset{\sim}{a}\right| + \frac{\sqrt{3}}{2} \times 200(\sqrt{6}-\sqrt{2}) - 100\sqrt{2} = 0$$

$$\frac{1}{\sqrt{2}}\left|\underset{\sim}{a}\right| = 100(\sqrt{18}-\sqrt{6}) - 100\sqrt{2}$$

$$\left|\underset{\sim}{a}\right| = 100\sqrt{2}(2\sqrt{2}-\sqrt{6})$$

$$\therefore \left|\underset{\sim}{a}\right| = 100(4-2\sqrt{3}) \quad N$$